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COMMENT

Comment on 'Integrable Kondo impurity in one-dimensional *q*-deformed *t–J* models'

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Abstract
The relation between integrable magnetic impurities located at the boundary and embedded into a one-dimensional correlated electron chain is discussed. Misleading statements by Ge *et al* are corrected and placed into proper context.

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In a recent paper, Ge *et al* [1] claim to have solved the boundary Kondo impurity problem in one-dimensional *q*-deformed *t–J* models. Impurities can either be placed at the boundary of a chain or be embedded into the chain. For the latter case, periodic or open boundary conditions yield the same results for the behaviour of the magnetic impurity itself. The condition of integrability imposes the absence of a reflection amplitude on impurities embedded into the host (elastic scatterers), while if placed at the boundary the impurity together with the edge are pure reflectors and have, by their nature, no transmission. The two cases can be comprised by considering an open-end chain and allowing the impurity to be placed anywhere along the chain [2]. If placed at the boundary the edge itself produces the reflection properties [2, 3]. This allows a direct comparison of the two situations. The properties of the impurity situated in the bulk and at the edge of the integrable *t–J* chain as a function of the magnetic field are the same [2]. Moreover, the structure of the Bethe ansatz equations for this case of open boundary condition (including the impurity factors) is identical in both cases of an impurity situated in the bulk of an open chain and at the edge of that chain.

An impurity embedded into a lattice interacts with the two neighbouring sites and is in this sense nonlocal (it involves three-site interactions). Consequently, the impurity interacts with two partial waves, e.g. right and left moving electrons, rather than one partial wave as for a contact or on-site interaction. An impurity at the edge is, by definition, the last site of the chain and has only one neighbour to interact with. In contrast to impurities at the boundary, a finite concentration of impurities can be embedded into a correlated electron chain.

In their paper, Ge *et al* [1] incorrectly criticized the method employed to solve the class of embedded impurity models. In this comment, we aim to correct these statements.

Their criticisms are quite general, affecting all magnetic impurities embedded into a Luttinger liquid, independently of the host being a strongly correlated electron system or a Heisenberg spin system. Their incorrect arguments are also independent of the type of impurity, whether Kondo-like or Anderson-like or any other impurity.

We believe that, while some of the arguments presented by Ge *et al* [1] are clearly false, other statements are implicitly incorrect. Below we respond to these issues (not in their order of appearance in [1] but in a logical order), so the reader can form his/her own opinion about this rather complicated and subtle subject.

(1) Contrary to what is claimed by Ge *et al* [1], the monodromy matrix is properly defined for each of the embedded impurity systems we consider. The monodromy matrix is usually denoted by $\hat{L}$ and is defined as the product of the $\hat{X}$ two-particle scattering matrices for each electron and one factor corresponding to the impurity, $\hat{S}$. For instance, for the supersymmetric t–J model with an embedded mixed valent impurity the monodromy is defined in equation (2.8) of [4], for an impurity of the Kondo exchange type embedded into the supersymmetric t–J model it is shown in appendix A as equation (A3) of [5], and for the degenerate Anderson impurity ($U \to \infty$ limit) embedded in the degenerate t–J model as equation (7) in [6]. The monodromy matrix has the same generic form for every integrable combination of correlated electron host and impurity. The $\hat{X}$ matrices for the host and $\hat{S}$ for the impurity differ from case to case. Both are explicitly defined in each of our papers. (For other combinations of host and impurity see [7, 8].)

(2) At this point we would like to stress the well-known fact that there are two different (independent) approaches to the algebraic Bethe ansatz for systems of particles with internal degrees of freedom: (i) that mentioned in point (1), which for impurity models has been extensively reviewed in [9] (the number of operators in the monodromy is given by the number of electrons and the impurity); and (ii) the graded approach, in which the charge sector is treated as one more degree of freedom, see, for example, [10, 11] (the number of operators in the monodromy is given by the number of sites plus impurity). For magnetic impurities in a correlated electron host this approach has been used in [2, 3, 12]. In each of these papers the monodromy is properly defined. This is also the case in [13, 14], which correspond to embedded nonmagnetic impurities.

(3) The condition of integrability requires that the $\hat{X}$ matrices satisfy the Yang–Baxter relations (also known as triangular relations) among themselves and with the impurity $\hat{S}$ matrix. For a given host, the choice of impurity is then not arbitrary. Note that the impurity $\hat{S}$ matrix is generally a two-parameter function (a discrete parameter is the spin of the impurity and the coupling to the host, θ, is a continuous parameter) and the triangular relation is satisfied for all values of the parameters. The triangular relation is also obeyed for $\hat{S}$ matrices with different values of θ, so that a finite concentration of impurities with a distribution of coupling constants can be embedded into a correlated electron host (or Heisenberg chain) without destroying the integrability.

Let us illustrate this with the example of the graded approach for the $su(1|2)$-symmetric t–J model with a dynamical magnetic impurity. In the graded approach one can start from the R-matrix $R = (u+i)^{-1}(iI + uP)$, where u is the spectral parameter, $I = \delta_{\alpha,\beta}\delta_{j,k}$ is the identity matrix ($\alpha, \beta, j, k = 1, 2, 3$ denote all possible states), and $P = (-1)^{[j][\alpha]}\delta_{\alpha,k}\delta_{j,\beta}$ is the graded permutation operator (e.g. FFB grading with $[1] = [2] = 1$ and $[3] = 0$). These R-matrices satisfy the standard Yang–Baxter relations

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u). \tag{1}$$

We introduce the usual L-operators for each site of the inhomogeneous lattice $L_j(u_j) = PR(u_j)$, where j denotes the quantum space pertaining to the Hilbert space of the jth site

of the chain and u_j ($j = 1, \ldots, N$, where N is the number of sites) are the rapidities of the inhomogeneous lattice [11]. The L-operator of the impurity for the simplest case of $S = 1/2$ (higher spin cases can be considered similarly) is defined as $L_{\rm imp} \equiv L(u-\theta)$, i.e. it differs from L-operators of other sites of the lattice by the shift of the spectral parameter. The constant θ defines the coupling of the impurity site to the host chain. Naturally, the L-operators (including the impurity one) satisfy the Yang–Baxter relations

$$R_{12}(u_1 - u_2)L_1(u_1)L_2(u_2) = L_2(u_2)L_1(u_1)R_{12}(u_1 - u_2). \tag{2}$$

The monodromy of the inhomogeneous chain for periodic boundary conditions can be defined as

$$\hat{L}^{PBC}(u, \theta, u_1, \ldots, u_N) = L_{\rm imp}(u - \theta)L_1(u - u_1) \cdots L_N(u - u_N). \tag{3}$$

These monodromies satisfy the Yang–Baxter relations

$$R_{12}(u - v)\hat{L}^{PBC}(u)\hat{L}^{PBC}(v) = \hat{L}^{PBC}(v)\hat{L}^{PBC}(u)R_{12}(u - v) \tag{4}$$

where only the dependence on spectral parameters is written explicitly. Obviously, this relation holds for the impurity L-operator embedded at any place of the monodromy operator (i.e. the impurity can be placed into the chain at any link), as well as without the impurity operator. The transfer matrices, defined as the traces over the auxiliary spaces of the monodromies, mutually commute for different spectral parameters, which constitutes the exact integrability of the problem.

For open boundary conditions one can introduce additional reflection matrices $K(u)$, which satisfy the reflection equations [15]

$$R_{12}(u - v)K_1(u)R_{21}(u + v)K_2(v) = K_2(v)R_{12}(u + v)K_1(u)R_{21}(u - v). \tag{5}$$

We define the monodromy for an open system (see [2, 3, 12])

$$\hat{L}^{OBC}(u, \theta, u_1, \ldots, u_N) = \hat{L}^{PBC}(u, \theta, u_1, \ldots, u_N)K(u)(\hat{L}^{PBC})^{-1}(-u, \theta, u_1, \ldots, u_N) \tag{6}$$

which satisfies the reflection equation

$$R_{12}(u - v)\hat{L}_1^{OBC}(u)R_{21}(u + v)\hat{L}_2^{OBC}(v) = \hat{L}_2^{OBC}(v)R_{12}(u + v)\hat{L}_1^{OBC}(u)R_{21}(u - v) \tag{7}$$

where the indices 1 and 2 explicitly denote the auxiliary subspaces for the monodromies. Clearly, the reflection equations for the monodromies with open boundary conditions are satisfied for the impurity L-operator placed in any order in the monodromy (i.e. the impurity can be situated at any link of the open chain, including exactly at the edge of the chain) [2, 3, 12]. We choose diagonal reflection matrices $K(u) = \delta_{j,k}h_j(u)$, where $j, k = 1, 2, 3$ and $h_j(u)$ are determined by the values of boundary potentials and/or magnetic fields. The transfer matrices for open chains, defined as the traces of the monodromies over the auxiliary spaces, mutually commute for different spectral parameters, as for the case of periodic boundary conditions. In this way the exact integrability is preserved. We emphasize that both Yang–Baxter relations and reflection equations need to be satisfied for the integrablity of an open chain for the non-trivial cases of more than one site in a chain.

(4) As a consequence of the impurity matrix $\hat{S}$ the components of the monodromy matrix, denoted by $\hat{A}_{11}$, $\hat{A}_{12}$, $\hat{A}_{21}$ and $\hat{A}_{22}$ by Ge *et al* [1] in their introduction, are different from those of the correlated electron chain without impurity. The statements of Ge *et al* appear to erroneously imply that these components do not change. The commutation relations of the $\hat{A}_{ij}$ operators, however, are indeed the same as for the chain without impurity within scheme (i) of the algebraic Bethe ansatz (see point (2)). This is the consequence of the Yang–Baxter relations obeyed by the $\hat{X}$ and $\hat{S}$ matrices and the underlying Yangian symmetry. This is different, however, within formulation (ii) (see point (2)), where the effect of the impurity

matrix in the charge sector (first level Bethe ansatz) changes the commutation relations in the spin sector (second level).

(5) The vacuum state of the correlated electron chain with impurity is the direct product of the vacuum state of the chain without impurity with the spin-polarized state of the impurity. The two vacuum states are therefore not identical. The operator $\hat{A}_{21}$ plays the role of a 'raising operator' and, when applied to the vacuum state, it yields zero. This is consistent because both the vacuum and the $\hat{A}_{ij}$ matrices are modified due to the presence of the impurity. In their statements, Ge *et al* seem to question this issue.

In all of our papers we have considered impurities having the same spin symmetry as the host (in most cases, it is $SU(2)$ symmetry). In the terminology used in [1], this corresponds to $l = 1$.

(6) The trace of the monodromy, $\hat{A}_{11} + \hat{A}_{22}$, is of course modified by the presence of the impurity, which is a consequence of the impurity $\hat{S}$ matrix factor in the monodromy matrix. Ge *et al* [1] seem to have misunderstood this point. It is also incorrect that impurities can only modify the first level Bethe ansatz equations. The second (and higher) level Bethe ansatz equations arise from eliminating 'unwanted' terms from the Bethe states. Whether the second level Bethe ansatz equations contain impurity factors depends on the kind of impurity considered. For instance, an Anderson impurity ($S = 1/2$) would act only on the first level Bethe ansatz equations, and the spin dynamics is generated through spin-paired electron states (charge two-strings) [6]. However, an impurity with an explicit Kondo exchange (see [5]) or an Anderson impurity with internal spin dynamics (mixed valent impurity between two magnetic configurations) [4, 7, 8] act on both first and second level Bethe ansatz equations. It is now absolutely clear from the above (see point (3)) that, for open boundary conditions, the Bethe ansatz equations are the same for a magnetic impurity situated at any link of the host (i.e. in the bulk or at edges).

(7) The trace of the monodromy matrix is the transfer matrix of the system. Transfer matrices for different spectral parameters commute with each other and generate the conserved currents of the system. The first derivative of the logarithm of the transfer matrix for periodic boundary conditions is usually the Hamiltonian, although higher derivatives could also be chosen. Although the procedure is very tedious, the Hamiltonian can be obtained this way. The construction of the Hamiltonian is only possible within the graded approach of the algebraic Bethe ansatz, i.e. the nested scheme (ii) (see point (2)). This has been carried out explicitly for the simpler situations of a magnetic impurity of arbitrary spin embedded into Heisenberg chains of host spin $1/2$ [16] and 1 [17]. The impurity is located on a link and interacts with both neighbouring host sites. (Note that for the Heisenberg chain both schemes (i) and (ii) for the algebraic Bethe ansatz coincide.) The situation is similar (only more involved) for a magnetic impurity embedded into a correlated electron system (for instance, the $SU(3)$ invariant t–J model is equivalent to a spin 1 Heisenberg chain). The generic exact form of the Hamiltonian has been derived in [2] (see equation (7)) and [3] (see equation (1)). The procedure has been reviewed in appendix B of [12]. (The schematic form of the impurity Hamiltonian is also given in [6] (see equations (14)–(16)), [5] (see equations (B8)–(B10)), [7] (see equation (9)) and [18] (see equation (9)).

(8) An integrable impurity embedded into a host lattice is located on a link of the chain and interacts with electrons on both sites joined by the link. Although the reflection amplitude is zero as a consequence of the integrability, the impurity interacts with both partial waves (forward and backward moving electrons). The interaction necessarily involves three sites (the impurity and two neighbouring sites of the chain) and is in this sense nonlocal. The same construction for an impurity at the edge yields a local interaction, i.e. with only one partial wave or site, due to the natural absence of a second site [3]. It is frequently overlooked that,

if the impurity is placed at the edge, the effects of the boundary fields (boundary fields and impurity yield additive contributions to the energy) have to be separated.

Below we state the Bethe ansatz equations arising from the two schemes of the algebraic Bethe ansatz described above (both schemes yield the same result), for the $su(1|2)$-symmetric t–J model with dynamical magnetic $S = 1/2$ impurity and open boundary conditions with one of the possible choices for the diagonal reflection matrix (with local boundary potentials $\mu^{\pm}$ acting only on the edges of the open t–J chain). The eigenfunctions and eigenvalues of the Schrödinger equation are parametrized by the charge rapidities $\{p_j\}_{j=1}^{N_e}$, where N_e is the number of electrons in the system, and the spin rapidities $\{\lambda_\alpha\}_{\alpha=1}^{M}$, where M is the number of down spins, which are solutions of the Bethe ansatz equations. We point out again that these equations do not depend on the position of the dynamical magnetic impurity, i.e. they are the same for the impurity situated at any link in the bulk or at any edge. The Bethe ansatz equations read (see [2, 3, 12])

$$\begin{aligned} \prod_{\pm} e_1(\lambda_\alpha \pm \theta) \prod_{j=1}^{N_e} e_1(\lambda_\alpha \pm p_j) &= \prod_{\pm} \prod_{\substack{\beta=1,\\ \beta\neq\alpha}}^{M} e_2(\lambda_\alpha \pm \lambda_\beta) \\ e_1^{2N}(p_j) \prod_{\pm} e_{\xi^{\pm}}(p_j) e_2(p_j \pm \theta) &= \prod_{\pm} \prod_{\beta=1}^{M} e_1(p_j \pm \lambda_\beta) \end{aligned} \tag{8}$$

where $e_n(x) = (2x + \mathrm{i}n)/(2x - \mathrm{i}n)$ and $\xi^{\pm} = 2(1 - 1/\mu^{\pm})$. Clearly, the choice of one of $\xi^{\pm}$ being equal to zero and the other to $-2\mathrm{i}\theta + 2$ [12] reproduces the Bethe ansatz equations of [1] for the isotropic case $\gamma \to 0$.

In summary, the general form of the Bethe ansatz equations is the same for both impurities embedded into the host or impurities placed at the boundary for open boundary conditions. The properties of the impurity are then independent of their position in the chain. The coupling of the impurity to the host is usually characterized by an 'impurity rapidity' θ. This impurity rapidity can be any complex number (this does not affect the integrability, i.e. the Yang–Baxter relations). If θ is real, the usual thermodynamic properties expected for the Kondo effect are obtained. An imaginary θ, on the other hand, yields a non-Hermitian 'Hamiltonian' with three-spin terms if the impurity is embedded in the lattice. This situation can be overcome [2, 3, 12] if the impurity is placed at the edge of an open chain, where the non-Hermitian three-site terms vanish. The Bethe ansatz equations are, however, the same in both cases of an impurity situated in the bulk of an open chain or at the edges of an open chain. An imaginary θ leads to a boundary bound state in the first level Bethe ansatz equation (charge sector). This bound state suppresses the Kondo effect and is interpreted as an 'image' or 'ghost' spin.

Unfortunately, Ge *et al* have not presented the physical consequences of their model constructions.

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